Subordination chains and univalence criteria (New Extension of Historical Theorems for Univalent Function Theory)

Pascu, Nicolae N.; Raducanu, Dorina; Owa, Shigeyoshi

数理解析研究所講究録
1164 巻 2000 年 111-117

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Nicolae N. Pascu, Dorina Răducanu and Shigeyoshi Owa

Abstract

The object of the present paper is to give an univalence condition for analytic functions in the open unit disk U by using the properties for subordination chain.

1 Introduction

Let U be the open unit disk in the complex plane $\mathbf{C}$, i.e. $U = \{z \in \mathbf{C} : |z| < 1\}$. We denote by A the class of functions $f(z)$ which are analytic in U with $f(0) = 0$, $f'(0) = 1$ and by S the subclass of the class A consisting of univalent functions.

If $f(z) \in A$ and $g(z) \in S$, then $f(z)$ is said to be *subordinate* to $g(z)$(written by $f(z) \prec g(z)$)in U if $f(U) \subset g(U)$.

A function $L : U \times [0, \infty) \to \mathbf{C}$ is said to be a *subordination chain* if $L(\cdot, t)$ is analytic and univalent in U, for all $t \in [0, \infty)$ and $L(z, s) \prec L(z, t)$, whenever $o \leq s \leq t < \infty$.

The following result concerning subordination chains is due to Ch. Pommerenke [3].

Theorem 1 *Let $L(z,t) = a_1(t) z + ...$be a function from $U \times [0, \infty)$ into $\mathbf{C}$, such that:*

Mathematics subject classification(1991): 30C45
Key words and phrases: Subordination, subordination chain, univalence.

(i) $L(\cdot,t)$ *is analytic in* U*, for all* $t \in [0,\infty)$.

(ii) $L(z,t)$ *is a locally absolutely continuous function of* t*, locally uniformly with respect to* $z \in U$.

(iii) $a_1(t) \neq 0$*, for all* $t \in [0,\infty)$ *and* $\lim_{t\to\infty} |a_1(t)| = \infty$.

(iv) the family of function $\left\{\frac{L(z,t)}{a_1(t)}\right\}_{t\geq 0}$ *forms a normal family in* U.

Let $p : U \times [0,\infty) \to \mathbf{C}$ *be an analytic function in* U *with* $\operatorname{Re} p(z,t) > 0$ *for all* $(z,t) \in U \times [0,\infty)$ *and such that:*

$$\frac{\partial L(z,t)}{\partial t} = zp(z,t)\frac{\partial L(z,t)}{\partial z}, \tag{1}$$

a.e. $t \in [0,\infty)$, *for all* $z \in U$.

Then the function $L(z,t)$ *is a subordination chain in* U.

2 Sufficient conditions for univalence

By using Theorem 1 we obtain an univalence condition which generalize some known univalence criteria for analytic functions in the open unit disk U.

Let $a(t)$ be a complex valued function on $[0,\infty)$ satisfying:

$$\bullet a \in C^1[0,\infty),\ a(0) = 1,\ a(t) \neq 0 \tag{2}$$

and

$$a(t) + a'(t) \neq 0,\ t \in [0,\infty),$$

•the modulus of $a(t)$ is increasing to ∞. (3)

Definition 1 *Let $F = F(u,v)$ be a function from $U \times \mathbf{C}$ into $\mathbf{C}$ and let $L(z,t) = F(e^{-t}z, a(t)z)$, for all $(z,t) \in U \times [0,\infty)$. We say that the function F satisfies (PA) conditions if:*

(i) $L(\cdot,t)$ is analytic in U, for all $t \in [0,\infty)$.

(ii) $L(z,t)$ is a locally absolutely continuous function of t, locally uniformly with respect to $z \in U$.

(iii) the function $\frac{\partial L(z,t)}{\partial t} / z\frac{\partial L(z,t)}{\partial t}$ is analytic in $\bar{U}$, for all $t > 0$ and is analytic in U for $t = 0$.

(iv) $\frac{\partial F(0,0)}{\partial \nu} \neq 0$ and $\frac{\partial F(0,0)}{\partial u} / \frac{\partial F(0,0)}{\partial \nu} \notin (-\infty,-1]$.

(v) the family of functions

$$\left\{F\left(e^{-t}z, a(t)z\right) / \left[e^{-t}\frac{\partial F(0,0)}{\partial u} + a(t)\frac{\partial F(0,0)}{\partial v}\right]\right\}_{t\geq 0}$$

is a normal family in U.

Theorem 2 *Let $a : [0,\infty) \to \mathbf{C}$ be a function satisfying (2) and (3). Further, suppose $F : U \times \mathbf{C}$ is a function which satisfies (PA) conditions. If*

$$\left|G(z,z) + \frac{a(t) - a'(t)}{2a(t)}\right| < \frac{|a(t) + a'(t)|}{2|a(t)|}, \quad z \in U, \quad t \geq 0 \tag{4}$$

and

$$\max_{|z|=e^{-t}} \left|G\left(z, a(t)\frac{z}{|z|}\right) + \frac{a(t) - a'(t)}{2a(t)}\right| \tag{5}$$

$$\leq \frac{|a(t) + a'(t)|}{2|a(t)|}, \quad z \in U \setminus \{0\}, \quad t \geq 0,$$

where

$$G(u,v) = \frac{u}{v} \cdot \frac{\partial F(u,v)}{\partial u} / \frac{\partial F(u,v)}{\partial v}, \tag{6}$$

then $F(z,z)$ is an univalent function in U.

Proof. We wish to show that the function $L(z,t) = F(e^{-t}z, a(t)z)$ satisfies the conditions of Theorem 1 and hence $L(\cdot,t)$ is univalent in U, for all $t \in [0,\infty)$.

If $F(e^{-t}z, a(t)z) = a_1(t)z + \ldots$, then

$$a_1(t) = e^{-t}\frac{\partial F(0,0)}{\partial u} + a(t)\frac{\partial F(0,0)}{\partial v}.$$

By using the conditions (iv) and (v) of the Definition 1 we have $a_1(t) \neq 0$ for all $t \geq 0$, $\lim_{t\to\infty} |a_1(t)| = \infty$ and the family of functions $\left\{\frac{L(z,t)}{a_1(t)}\right\}_{t\geq 0}$ is a normal family in U. Let

$$p(z,t) = \frac{\partial L(z,t)}{\partial t} / z\frac{\partial L(z,t)}{\partial z}, \quad (z,t) \in U \times [0,\infty). \tag{7}$$

Then the condition (1) of Theorem 1 is satisfied for all $z \in U$ and $t \in [0,\infty)$. It remains to prove that the function $p(z,t)$ has a positive real part in U, for all $t \in [0,\infty)$. If

$$w(z,t) = \frac{1 - p(z,t)}{1 + p(z,t)}, \quad (z,t) \in U \times [0,\infty), \tag{8}$$

then $\mathrm{Re}\, p(z,t) > 0$ if and only if $|w(z,t)| < 1$. According with (6), (7), (8) we have

$$w(z,t) = \frac{2a(t)}{a(t) + a'(t)} G\left(e^{-t}z, a(t)z\right) \tag{9}$$

$$+\frac{a(t) - a'(t)}{a(t) + a'(t)}, \quad (z,t) \in U \times [0,\infty).$$

By using the inequality (4) we obtain $|w(z,0)| < 1$, for all $z \in U$. For $t > 0$ the function $p(z,t)$ is analytic in $\bar{U}$ and it follows

$$|w(z,t)| < \max_{|\zeta|=1} |w(\varsigma,t)| = \max_{|\zeta|=1} \left| \frac{2a(t)}{a(t) + a'(t)} G\left(e^{-t}\varsigma, a(t)\varsigma\right) + \frac{a(t) - a'(t)}{a(t) + a'(t)} \right|.$$

If we let $z = e^{-t}\varsigma$ with $|\zeta| = 1$, then $|z| = e^{-t}$ and by using (5) we have

$$|w(z,t)| < \max_{|z|=e^{-t}} \left| \frac{2a(t)}{a(t)+a'(t)} G\left(z, a(t)\frac{z}{|z|}\right) + \frac{a(t)-a'(t)}{a(t)+a'(t)} \right| \leq 1.$$

Since $L(z,t)$ satisfies all the conditions of Theorem 1, it follows that $L(z,t)$ is a subordination chain in U and $F(z,z) = L(z,0)$ is an univalent function in U.

Remark 1

1. If $a(t) = e^t$ we obtain the univalence condition due to N.N. Pascu [1].

2. If
$$F(u,v) = f(u) + \frac{(v-u)R(u)}{1-(v-u)Q(u)},$$
where $u = e^{-t}z, v = a(t)z$ and $R(z), Q(z)$are analytic functions in U, we obtain the results concerning univalence criteria due to J.A. Pfaltzgraff [2].

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Nicolae N. Pascu , Dorina Răducanu:
Department of Mathematics
Faculty of Sciences, Transilvania University
Iuliu Maniu, 50
Braşov 2200
România

Shigeyoshi Owa:
Department of Mathematics
Kinki University
Higashi - Osaka, Osaka 577-8502
Japan